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DYNAMIC RESPONSE OF PULSE-LOADED RATE-SENSITIVE STRUCTURES

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Abstract—In a recent publication [1] a simplified general method was developed to determine the response of impulsively loaded, perfectly plastic, rate-sensitive structures with highly non-linear yield stress–strain rate laws.

In the present paper this approach is extended to encompass the dynamic load application phase. In other terms, a mathematical model closer to the physical situation is considered, namely, a pressure loaded rather than an impulsively loaded structural element.

As the prototype of a general structure the dynamically loaded thin ring is considered. For a constant pressure pulse, exact and approximate responses are calculated and differ by only a few percent. In the approximate solution the yield stress is taken to be a constant associated with the peak strain rate (which occurs at the instant of load termination).

The results suggest that when a dominant plastic mode of deformation exists, one should be able to determine the response of a structure with arbitrary loaded pulse by enforcing an impulse/momentum change condition, while neglecting forces arising owing to deformation, to determine the initial strain rate distribution. The associated yield stress distribution may be calculated and assumed to be constant at each field point during the entire motion.

INTRODUCTION

THE dynamic plastic response of rate-sensitive structures has been the subject of numerous experimental and analytical studies in recent years. Despite the significant increasing effort in this area of structural behavior, however, workable engineering methods for realistic structures are not yet forthcoming.

At the root of the difficulty lies a mathematical problem: the strong nonlinearity of the plastic rate-sensitive material response. Although most of the effort to date has been focused on the simpler perfectly plastic rather than strain hardening materials, this nonlinearity has still been a genuine stumbling block to progress in this area.

In a series of studies on dynamically loaded cantilever beams, reasonable comparisons were effected between experiments and perfectly plastic rate-sensitive theoretical analyses [2–5]. Unfortunately, the analyses are necessarily complex and not readily extended to more complex structures. A lumped mass–finite difference technique was successfully employed by Pian and his associates [6], but this approach is numerically tedious and not likely to be practical for large realistic structures.

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The obvious need for accurate approximate methods of analysis has prompted renewed efforts in this direction [1, 7, 8]. When most of the plastic response occurs in a given mode of deformation, Symonds suggests a uniform rate–sensitivity correction should suffice [7]. Concentrating on impulsively loaded rings and rods, Perrone has shown that good approximations to the "exact" response are possible by assuming a constant dynamic yield stress associated with the maximum strain rate [1]. Extensions to strain hardening rate-sensitive behavior also appear feasible [8].

In the present paper the analysis discussed in [1] for impulsively loaded, perfectly plastic structures will be extended to apply to finite pressure pulse loaded structures. Accordingly, the change of yield stress with strain rate during the loading phase must be accounted for carefully.

In the next section the "exact" solution of a rate-sensitive ring under constant pressure pulse is determined. Two different types of approximate solutions are discussed in the following two sections; all solutions are compared with respect to their relative utility in the next section. Conclusions are drawn in the final section.

COMPLETE SOLUTION TO RING UNDER PRESSURE PULSE

The essential complication associated with plastic rate-sensitive structural response is the nonlinear variation of strength properties with strain rate; as such, it is most appropriate, when developing analytical techniques, to investigate structural elements that are realistic enough to be tested in the laboratory, and yet simple enough not to introduce additional complexities, (e.g. bending, geometric complications), that would becloud rate-sensitivity aspects of the response. In this connection, the ring problem appears to offer an ideal compromise as a structural element.

For this problem, precise operational laboratory techniques are already available [9]. In addition, exact and accurate approximate solutions have been obtained to the impulsively loaded rate-sensitive ring [1].

In the present paper, exact and accurate simplified approximate solutions are calculated for the pressure pulse loaded ring, Fig. 1. The solution may be delineated into two parts: a loading or acceleration phase, and an unloading or deceleration phase. Peak strain rate

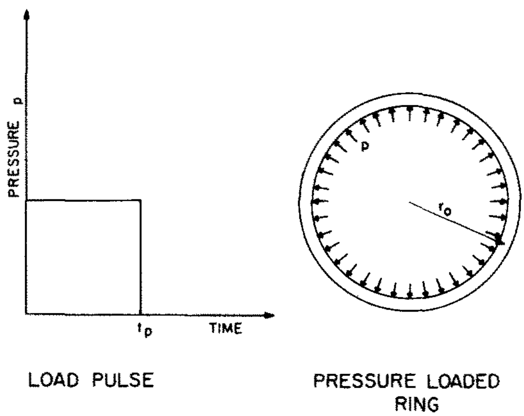

FIG. 1. Pressure pulse loaded ring.

is achieved at load termination. Actually, with but slight modification, the impulsively loaded ring solution [1] will apply in the second phase so that the essential problem will be to find the loading phase solution.

To begin with, let us enumerate the important assumptions made in developing the equations of motion: strain hardening, wave propagation effects and radial stress are ignored; the circumferential stress is uniform through the ring; the ring material is perfectly plastic and rate-sensitive as described by Fig. 2; relatively large deformations are permitted.

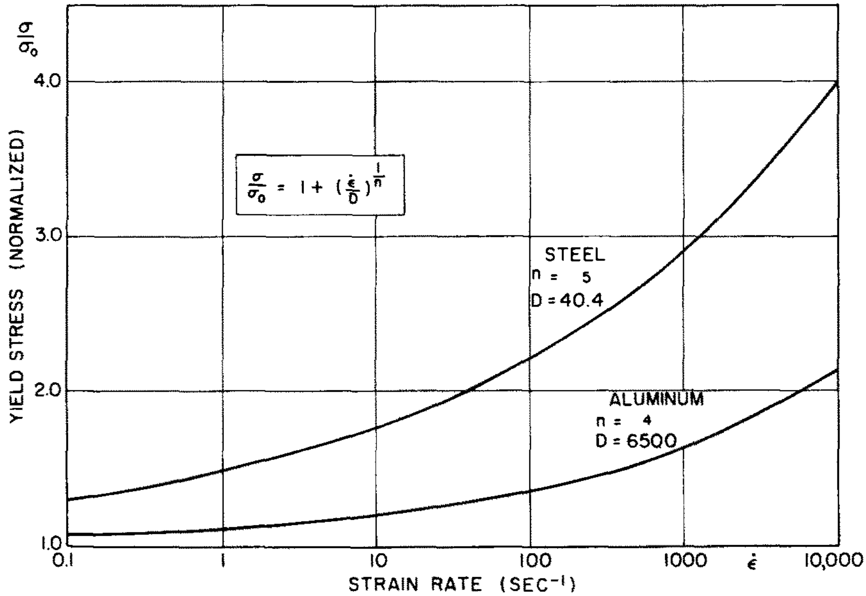

FIG. 2. Yield stress–strain rate law.

The definition of most of the variables of interest are collected and listed as follows:

u radial velocity of the ring
u_p peak velocity
r_0 initial ring radius
$\bar{r}$ current radius of the ring
t time
t_p time needed to reach the peak velocity
h ring depth
A_0 initial cross-section of the ring
A cross-section at any time t
σ_0 static yield stress of the ring material
σ dynamic yield stress
p_0 maximum static pressure which the ring can carry ($p_0 = \sigma_0 A_0/r_0 h$)
p constant pressure applied to the ring
ρ mass density of ring material
$\dot{\varepsilon}$ rate of strain ($\dot{\varepsilon} = u/\bar{r}$)

The governing equation of motion is derived by applying Newton's Second Law to a generic element of the ring

$$\rho A r_0 \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = -\sigma A + p\bar{r}h. \tag{1}$$

As noted earlier, σ, the circumferential stress varies with strain rate in accordance with the following expression.

$$\frac{\sigma}{\sigma_0} = 1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{\frac{1}{n}} \tag{2}$$

where D and n are material constants.

Substitution of equation (2) into equation (1) yields the result

$$\rho A_0 r_0 \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = -\sigma_0 A\left[1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{\frac{1}{n}}\right] + p\bar{r}h. \tag{3}$$

Since the material of the ring is incompressible

$$A_0 r_0 = A\bar{r}$$

and equation (3) becomes

$$\rho\bar{r}\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = -\sigma_0\left[1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{\frac{1}{n}}\right] + \frac{p\bar{r}}{A}h. \tag{4}$$

Equation (4) is the general differential equation of motion of the ring.

The foregoing equation can be solved more conveniently by choosing $\bar{r}$ as the independent variable and u as the dependent variable. This is the case because the initial and final conditions of the motion may be expressed more readily in terms of $\bar{r}$ and u. To effect this variable change the following relation is used

$$\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}(u^2)}{\mathrm{d}\bar{r}}.$$

As a result equation (4) assumes the following form

$$\frac{1}{2}\rho\bar{r}\frac{\mathrm{d}(u^2)}{\mathrm{d}\bar{r}} = -\sigma_0\left[1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{\frac{1}{n}}\right] + \frac{p\bar{r}}{A}h. \tag{5}$$

It will prove convenient to work with non-dimensional quantities. To this end the following dimensionless variables are defined:

$$v = u^2/u_p^2 \qquad \alpha = 2\sigma_0/(\rho u_0^2)$$

$$r = \bar{r}/r_0 \qquad \beta = u_p/(r_0 D)$$

$$P = p/p_0 \qquad \gamma = \alpha(\beta)^{\frac{1}{n}}$$

Consequently, equation (5) may be expressed as

$$\frac{dv}{dr} = -\frac{\alpha}{r} - \frac{\gamma}{r^{\left(1+\frac{1}{n}\right)}} v^{\frac{1}{2n}} + \alpha r P. \tag{6}$$

Further useful simplifications are effected via the variable change

$$r = 1+x$$

where x is the ratio of the increment of ring radius to the original radius.

Substituting into equation (6) while restricting the size of deformations such that

$$x \ll 1$$

we find the following result

$$\frac{dv}{dx} = \alpha(P-1) - \gamma v^{\frac{1}{2n}}. \tag{7}$$

Subject to the appropriate initial condition

$$v(0) = 0$$

the following solution is obtained to equation (7)

$$x = \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{[\gamma(p-1)]^k} \frac{v^{\frac{k-1}{2n}+1}}{[(k-1)/2n]+1}. \tag{8}$$

Of course the foregoing solution is applicable only during the loading phase, that is

$$0 \le x \le x_p$$

The ring response for the deceleration phase is readily determined using the solution given in Ref. [1] with trivial modification. The solution cited applies for an impulsively loaded ring with no initial displacement. For present purposes the only slight difference is that the ring has an initial displacement (corresponding to the deformation attained at the instant the load is removed). This difference is accounted for by modifying the initial condition in the solution of Ref. [1]. Hence, we have now obtained the complete deformation response of a rate-sensitive ring to a constant load pulse.

Representative numerical solutions are presented in Table 1 for the following choice of parameters:

$$n = 5 \qquad \alpha = 100, 25, 25$$

$$D = 40{\cdot}4/\text{sec} \qquad \gamma = 100, 34{\cdot}5, 50$$

$$P = 5, 10$$

Deformation of the ring at the end of the load phase is x_p, and the final total deformation is x_f. Simultaneous values of P, α and γ describe the load pulse.

APPROXIMATE SOLUTION I: CONSTANT YIELD STRESS ASSOCIATED WITH PEAK STRAIN-RATE

Initially, we shall examine from a qualitative viewpoint the general characteristics of the two phases of motion of the ring response. The motivation behind the selection of a particularly useful approximate approach should then become more apparent.

Experiments have been successfully conducted on dynamically loaded rings with impulsive type loading [9]. A typical deformation–time response curve from these tests is reproduced in Fig. 3. For a ring load with a finite pressure pulse the curve given in Fig. 3 may be interpreted as characteristic of the deceleration phase. In other terms, the origin of the above curve would correspond to the time when the finite pulse terminates.

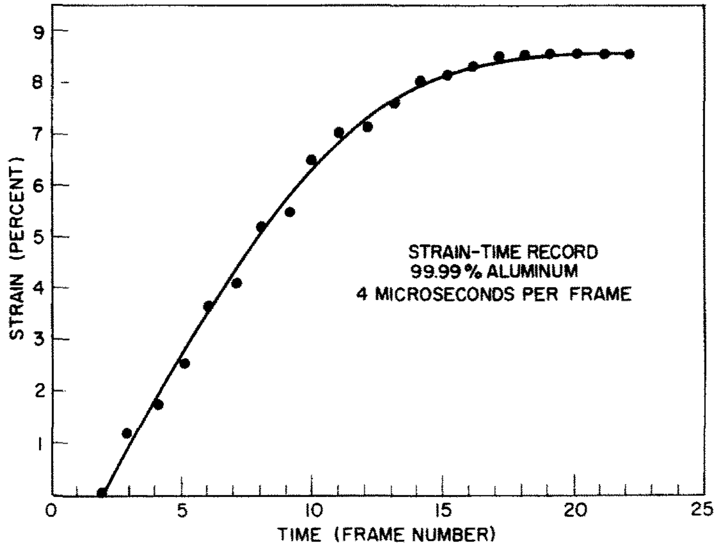

FIG. 3. Response of impulsively loaded ring [9].

A reasonable estimate of the response during the load phase may be obtained by using the leading (and also dominant) term of equation (8) in the equation of motion and integrating for a constant pressure pulse. The resulting deformation is a quadratic function of time (x–t curve is a parabola). Realizing that the velocity and displacement are initially zero and must be continuous at the interface between both phases, we deduce that the deformation response history should be of the form shown in Fig. 4a.

Obviously, during the bulk of the flow process, i.e. between B and D, the velocity, (Fig. 4c) which is the slope of Fig. 4a does not change appreciably. Consequently, over the same range the strain rate and hence, yield stress is substantially constant (Fig. 4b). Indeed, we may assume to good approximation that the yield stress is a constant corresponding to the peak strain rate, for the entire deformation range. This is equivalent to saying the area under the curve ABCDE in Fig. 4b, which is associated with plastic work absorbed by the ring is approximately equal to the area under the curve FCG. Consideration of a few numerical examples will verify the validity of this tentative hypothesis. It is worth noting

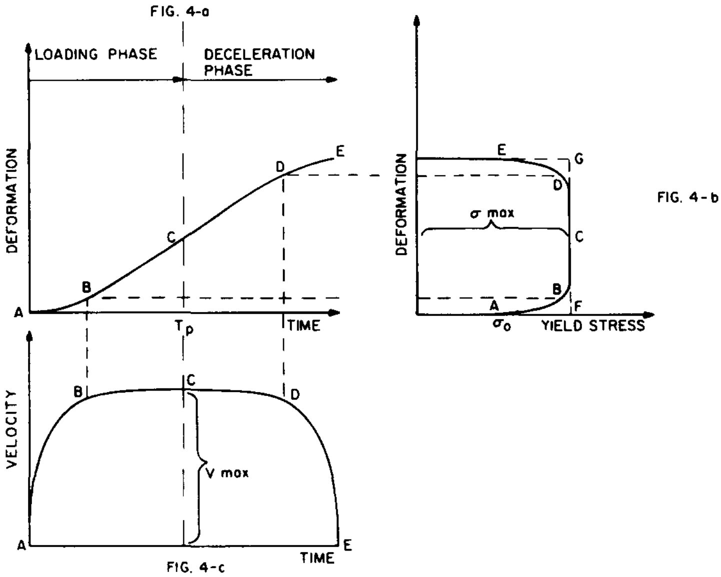

FIG. 4. Typical dynamic response curves.

that a similar approach was successfully utilized to determine the response of impulsively load rings [1]. In that study, of course, there was no acceleration or loading phase.

Let us turn our attention to the implementation of the suggested approximation for the loading phase. The yield stress is chosen to be a constant associated with the peak strain rate by simply rewriting the differential equation of motion equation (7) and replacing the term v on the right hand side by its peak value, namely unity.

For a constant pressure pulse the resulting differential equation may be trivially integrated with an initial zero velocity condition to find

$$x'_p = 1/[\alpha(P-1)-\gamma]. \tag{9}$$

This approximate solution is only applicable during the loading phase, i.e.

$$0 \leq x \leq x'_p$$

Primed values will be used to denote approximate solutions.

The solution for the deceleration phase is similar to the approximate solution obtained in Ref. [1]. The approximate total final deformation is termed x'_f.

To test the validity of the assumption of constant yield stress, numerical solutions obtained utilizing equation (9) are compared with the previous exact solutions presented in Table 1. The comparisons are shown in Table 2. It is very clear from Table 2 that the differences between the exact and approximate deformations are of the order of a few percent. Evidently, these results verify the assumption that most of the deformation takes place under substantially constant yield stress.

To illustrate an important point we now study more carefully the difference between the exact and approximate solutions for one of the foregoing cases, viz., $\dot{\varepsilon}_p = 40{\cdot}4$, $\alpha = \gamma = 100$, $P = 10$. We calculate the approximate deformation, x'_p, at the end of the loading phase from two viewpoints: the yield stress is taken to be constant and equal to

TABLE 1. EXACT RING RESPONSE $n = 5$, D = 40·4/sec

P	α	γ	$\dot{\varepsilon}_p$	σ/σ_0	x_p	x_f
5	100·0	100·0	40·4	2·00	0·00316	0·00829
5	25·0	34·5	202·0	2·38	0·01462	0·03211
5	25·0	50·0	1293·0	3·00	0·01843	0·03228
10	100·0	100·0	40·4	2·00	0·00123	0·00636
10	25	34·5	202·0	2·38	0·00516	0·02265
10	25	50·0	1293·0	3·00	0·00546	0·01931

$\dot{\varepsilon}_p$ = peak strain rate. x_p = deformation at load cut-off. x_f = final total deformation.

TABLE 2. COMPARISON OF EXACT SOLUTION AND APPROXIMATE SOLUTION I

P	α	γ	$\dot{\varepsilon}_p$	x_p	x'_p	x_f	x'_f
5	100	100·0	40·4	0·003164	0·00333	0·00829	0·00834
5	25	34·5	202·0	0·01462	0·01528	0·03211	0·03222
5	25	50·0	1293·0	0·01843	0·0200	0·03228	0·03342
10	100	100·0	40·4	0·00123	0·00125	0·00636	0·00626
10	25	34·5	202·0	0·00516	0·00525	0·02265	0·02219
10	25	50·0	1293·0	0·00546	0·00572	0·01931	0·01957

x_p = exact deformation at load cut-off. x'_p = approximate deformation at load cut-off (from solution I).
x_f = exact final total deformation. x'_f = approximate final total deformation (from solution I).

either its initial static value or its peak dynamic value. The results, along with the exact solution, for comparison are displayed in Fig. 5. Obviously, the results obtained for constant peak yield stress compare much more favorably with the exact solution than those associated with constant static yield stress. These results reinforce further the approximate technique which has been selected.

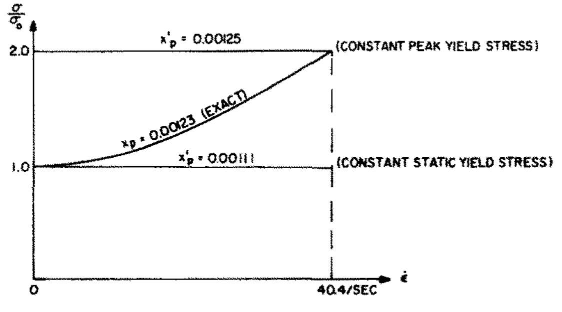

FIG. 5. Comparison of load phase responses for $\alpha = \gamma = 100$, $P = 10$.

By this time, a paradox may have been noticed by the discerning reader. This anomaly is concerned with the fact that the approximate deformation at load cut-off is higher (than its true or exact value) when we assume the yield stress constant and equal to its maximum value. This observation disturbs the physical intuition since the ring made of the stronger material should certainly deform a lesser amount.

As we shall see, the reason for this apparent discrepancy is the slight difference in pressure pulse length between both approaches. Earlier, for mathematical convenience, we elected to use deformation rather than time as the independent variable. The end of the load phase was designated as the instant when the dimensionless velocity attained unit value. As a result, we were not able to require that the pressure pulse length be uniform for the exact and approximate solutions. However, we will proceed to prove for the numerical case under discussion, that the pulse lengths for the exact and constant peak yield stress cases agree within 2 percent. The pulse length associated with the exact solution is found by rewriting the equation of motion [equation (7)], reverting back to time as the independent variable, and integrating to find

$$(t_p)_{exact} = \frac{2r_0}{u_0}\int_0^1 \frac{d(u/u_p)}{\alpha(P-1)-\gamma(u/u_p)^{\frac{1}{n}}}. \tag{10a}$$

For the constant peak yield stress case the pulse length is determined by integrating equation (7) again with time as the independent variable and with the term $v\,(= u^2/u_p^2)$ on the right hand side set equal to unity.

$$t'_p = \frac{2r_0}{u_0}\sqrt{\left[\frac{x'_p}{\alpha(P-1)-\gamma}\right]}. \tag{10b}$$

For the numerical case under discussion $\alpha = \gamma = 100$, $P = 10$ the foregoing time pulses assume the following values:

$$t_p = \frac{2r_0}{u_0}(0{\cdot}001225)$$

$$t'_p = \frac{2r_0}{u_0}(0{\cdot}00125).$$

Obviously, the time pulse for the constant peak yield stress case is 2 percent longer than for the exact case. Adjustment of the constant peak yield stress pulse length so that it coincides with the exact one results in an exact deformation which is only 2 percent higher than the approximate one.

The essential conclusions are, (a), the anomaly is explained and, (b), no serious error arises by not explicitly considering pressure pulse lengths.

APPROXIMATE SOLUTION II: CONSTANT YIELD STRESS VIA MOMENTUM IMPULSE

In the previous section a very good approximation to the response of a ring with a rectangular pressure pulse is obtained by taking the yield stress to be a constant associated with the peak strain rate. The simplifications attendant with this problem are such that it is a trivial task to determine the peak strain rate once the governing parameters of the problem are prescribed. On the other hand for structures with varying load pulses, the same situation does not prevail, and it behooves us to develop a more efficient alternate approximate approach.

As observed earlier, the variation of yield stress with strain rate is very slow (see Fig. 2). Moreover, it has been clearly established that most of the plastic flow occurs at the yield stress corresponding to the peak strain rate. Hence, an obvious alternative would be to estimate in some way the peak strain rate, determine the associated yield stress, take this yield stress to be a constant and proceed to solve as before.

To estimate the peak strain rate, we assume that the pressure pulse is applied impulsively. This is roughly equivalent to saying that in the loading phase the ring hoop stress is not a dominant factor in the ring response. Let us apply this approximate impulsive approach, initially for a constant and subsequently for an arbitrary pressure pulse.

The equation of impulse for an element of the ring may be expressed as follows:

$$F\,\mathrm{d}t = m\,\mathrm{d}u$$

where $(F\,\mathrm{d}t)$ represents the impulse itself and $(m\,\mathrm{d}u)$ the change in momentum. Substituting for F and m their respective values in the equation of impulse we find:

$$ph\,\mathrm{d}t = \rho A_0\,\mathrm{d}u.$$

Integration of the foregoing equation yields the result

$$u = \frac{ph}{\rho A_0} t. \tag{11}$$

Obviously, the peak velocity is given as

$$u_{\mathrm{p}} = \frac{ph}{\rho A_0} t_{\mathrm{p}}.$$

If we now non-dimensionalize with respect to p we find

$$u_{\mathrm{p}} = \frac{P\sigma_0}{\rho r_0} t_{\mathrm{p}} \tag{12}$$

and dividing by r_0 we obtain the peak strain rate

$$\dot{\varepsilon}_{\mathrm{p}} = \frac{\sigma_0}{\rho r_0^2} (Pt_{\mathrm{p}}). \tag{13}$$

For convenience let

$$C = \sigma_0/(\rho r_0^2)$$

realizing C is constant for any ring, then equation (13) becomes

$$\dot{\varepsilon}_{\mathrm{p}} = CPt_{\mathrm{p}}. \tag{14}$$

Obviously, the strain rate is directly propositional to the quantity (Pt), the area under the pressure–time curve.

Equation (14) can be generalized for any kind of arbitrary pressure distribution without any difficulty

$$\dot{\varepsilon}_{\mathrm{p}} = C \int_0^{t_{\mathrm{p}}} P\,\mathrm{d}t. \tag{15}$$

We can conclude from equation (15) that the approximate maximum strain rate is directly proportional to the area under the pressure–time distribution. Therefore once the distribution is given, $\dot{\varepsilon}_p$ can be estimated easily. From equation (2) σ_p (the yield stress when $t = t_p$) can be obtained and substituted into the basic differential equation of motion, equation (7). The solution will be similar to the one obtained in equation (9).

By utilizing the load cut-off time t_p calculated in the previous section, an estimate of the peak strain rate ($\dot{\varepsilon}_p$ impulse) is immediately forthcoming from equation (14).

A complete tally of all numerical results is displayed in Table 3. In view of the slow variation of yield stress with strain rate, the estimates of the strain rates by the momentum–impulse approach are obviously adequate. Indeed, the excellent comparison between exact and approximate deflections attests to this fact.

COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS

In the preceding sections solutions are obtained to the pressure pulse loaded ring utilizing a very accurate analysis (which we call "exact") and two approximate analyses (termed, for brevity, peak strain rate and momentum–impulse approximations).

Both approximate solutions employ a constant yield stress which unlike the exact case, does not vary with strain rate. With the peak strain rate approximation the yield stress is given the constant value associated with its peak strain rate; with the momentum–impulse approximation the yield stress is assigned a constant value corresponding to the strain rate calculated by assuming the load to be impulsively applied.

It is clear from Table 3 that the exact and approximate solutions of the ring problem differ by only a few percent. This observation is true for a wide variety of combinations of

TABLE 3. COMPARISON OF EXACT AND BOTH APPROXIMATE SOLUTIONS (I AND II)

P	α	γ	$\dot{\varepsilon}_{p(exact)}$	$\dot{\varepsilon}'_{p(II)}$	$x_{p(exact)}$	$x'_{p(I)}$	$x'_{p(II)}$	$x_{f(exact)}$	$x'_{f(I)}$	$x'_{f(II)}$
5	100	100·0	40·4	50	0·00316	0·00333	0·00342	0·00829	0·00834	0·00833
5	25	34·5	202·0	250	0·01462	0·01528	0·01564	0·03211	0·03222	0·03214
5	25	50·0	1293·0	1625	0·01843	0·0200	0·02084	0·03228	0·03342	0·03404
10	100	100·0	40·4	44	0·00123	0·00125	0·00125	0·00636	0·00626	0·00621
10	25	34·5	202·0	222	0·00516	0·00525	0·00527	0·02265	0·02219	0·02199
10	25	50·0	1293·0	1443	0·00546	0·00572	0·00575	0·01931	0·01957	0·01927

Approximate solutions denoted by primed quantities. p = subscripts refer to end of load phase. f = subscripts refer to final deformations. (I), (II) refer to approximate solutions.

variables. Differences between the solutions decrease as the ratio of the applied pressure p to the static pressure p_0 increases. These results are to be expected since by increasing p we approach the impulsive situation.

All the numerical results presented in Tables 2 and 3 are plotted for comparison in Figs. 6 and 7. Exact values are plotted on the ordinate and approximate values on the abscissa. Obviously, the extent to which the approximate values cluster around a 45° line is an index of the accuracy of the approximate solutions. Evidently, both approximations offer very good comparisons with exact solutions. Moreover, there is no discernible difference between the relative accuracy of the two approximations, i.e. both are equally good.

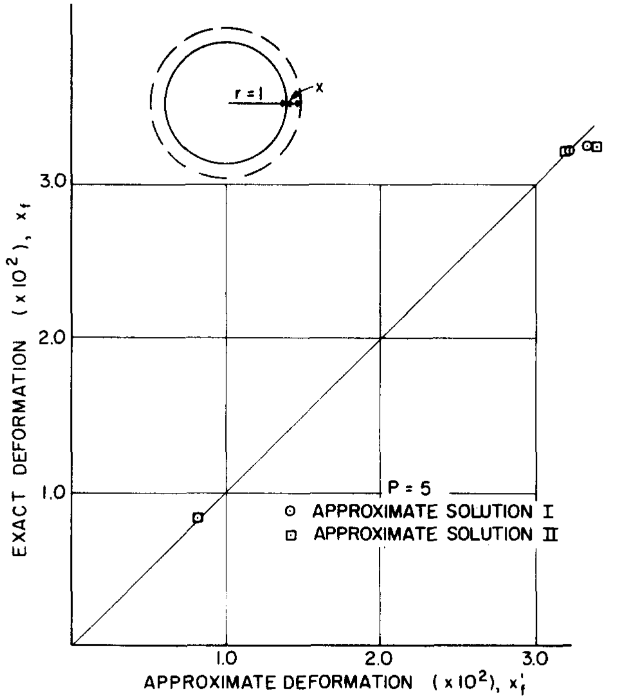

FIG. 6. Comparison of exact and approximate final deformations.

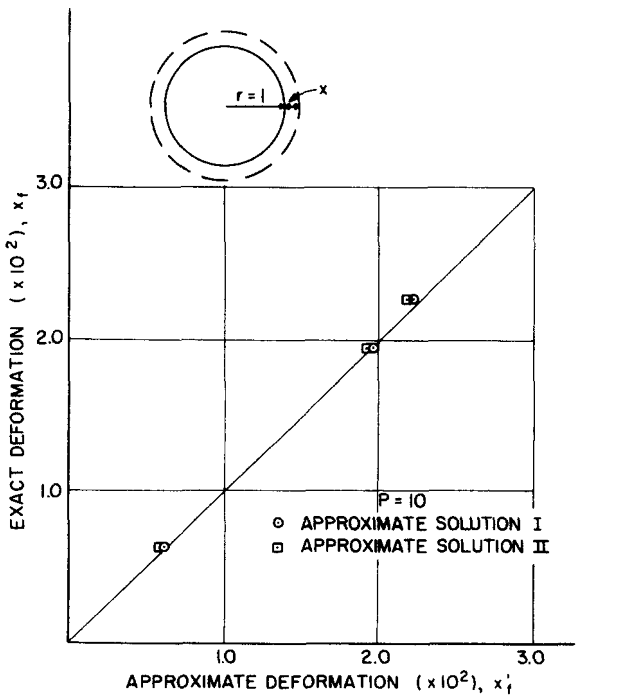

FIG. 7. Comparison of exact and approximate final deformations.

With respect to the difficulty and growth potential of the three solutions it is quite clear that the exact approach has the lowest prospects as it is the most difficult to apply and could be expected to get appreciably more difficult for complex realistic structures and loads. Although the peak strain rate approximation represents a significant simplification over the exact approach it is seriously limited by the difficulty in practical problems of determining the peak strain rate where the pressure is not constant. The momentum impulse approach clearly holds the greatest promise for eventual practical application.

DISCUSSION

In the present paper exact and approximate solutions are obtained to a pressure pulse loaded ring made of a perfectly plastic, nonlinear rate-sensitive material.

The essential assumption of both approximate solutions is that the bulk of the plastic flow for both phases of motion, (loading acceleration phase and unloading deceleration phase), occurs at a substantially constant yield stress associated with the maximum or peak strain rate. The second approximate solution differs from the first in that the peak strain rate yield stress, rather than being rigorously calculated, is only estimated via a momentum–impulse approach.

Both approximate approaches offer very fine numerical comparisons with exact solutions, deviations being of the order of a few per cent.

The momentum–impulse approach clearly merits the greatest attention for consideration of more complex structures, (e.g. variable pressure–pulse load, strain hardened plates or shells). To cite a specific example, it would appear quite straightforward to extend a recent impulsively loaded rate sensitive plate solution [10] to account for a finite pressure pulse load by this technique.

It is likely that the momentum/impulse approach would apply for problems such as discussed in Ref. [7] wherein most of the plastic response occurs in a given deformation mode.

When we reflect to consider that in addition to the rate-sensitive plastic structure analysis cited here, temperature-dependent yield structures are readily accounted for [11], it becomes apparent that we are rapidly approaching the threshold wherein practical engineering problems such as hypervelocity impact and explosive metal forming will submit to rational analytical study.

We end on a note of caution. The treatment presented here is contingent upon a perfectly plastic rate-sensitive power law, such as shown in Fig. 2. However, strain hardening behavior should require but a simple extension of the present results [8]. Experimental evidence to date has shown that many, if not most, structural metals do satisfy such material behavior laws.

Of course, plastic action is presumed to dominate over elastic action; and geometric non-linearities which may be significant in some practical applications are ignored.

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Абстракт—В предыдущей работе (1), выводится сокращенный, общий метод для определения реакций внезапно нагруженных, идеально пластических, чувствительных к скорости деформации конструкций, облающих высоко нелинейными законами напряжение течения—скорость деформации.

Внастоящей работе развивается этот подход с целью описания фазы применения динамической нагрузки. Иначе говоря, исследуется математическая модель, близкая физическому явлению, а именно, элемент конструкции нагружен давлением а не импульсной нагрузкой.

В качестве образца общей конструкции исследуется тонкое кольцо, нагруженное динамически. Определяются строгие и приближенные реакции для постоянной пульсации давления. Разница между ними составляет только несколько процентов. В приближенном решении напряжение текучести является постоянным, связанным с пиковой скоростью деформации /пто возникает при моменте окончания нагружения/.

Результаты указывают на то, что в случае когда действует преобладающая пластическая фрома дефромации тогда можно определить реакцию конструкции, загруженной произвольной пульсацией путем повышения этого импульса—условия изменения количества движения, так когда при пренибрежении сил, рост которых вызывает деформации, является возможным определить начальное распределение скорости деформации. Можно определить также связанное распределение напряжения текучести, которое является постоянным в каждой точке поля во время движения всех точек.